A note on the Swiss Solvency Test risk measure

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Abstract

In this paper we examine whether the Swiss Solvency Test risk measure is a coherent measure of risk as introduced in Artzner et al. [Artzner, P., Delbaen, F., Eber, J.M., Heath, D., 1999. Coherent measures of risk. Math. Finance 9, 203–228; Artzner, P., Delbaen, F., Eber, J.M., Heath, D., Ku, H., 2004. Coherent multiperiod risk adjusted values and Bellman's principle. Working Paper. ETH Zurich]. We provide a simple example which shows that it does not satisfy the axiom of monotonicity. We then find, as a monotonic alternative, the greatest coherent risk measure which is majorized by the Swiss Solvency Test risk measure.
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1. Introduction

With the enactment of the new Insurance Supervision Act on 1 January 2006 in Switzerland, a new risk-based solvency standard was set for the Swiss insurance industry. At its core lies the Swiss Solvency Test (SST), which is a principles-based framework for the determination of the solvency capital requirement for an insurance company. For more background on the SST we refer to the official web page of the Swiss Federal Office of Private Insurance – Topics – SST (Swiss Federal Office of the Private Insurance, 2007).

Part of the SST framework is the SST risk measure ρ_{SST} which assigns a capital requirement (target capital) to the run-off of the in force asset–liability portfolio. In this sense, ρ_{SST} is a multiperiod risk measure. At the same time, there has been a well-established axiomatic theory of risk measures in the financial and insurance mathematics literature, see e.g. Artzner et al. (1999), Denuit et al. (2006), Föllmer and Schied (2004), Kaas et al. (2001), Artzner et al. (2004) and Cheridito et al. (2006) for the multiperiod case. In this context, it is worth mentioning that "optimal", not necessarily coherent, assessment of risk capital is still the subject of on-going

research, see for instance Danielsson et al. (2007), Dhaene et al. (in press), Föllmer and Schied (2002) and Rootzen and Klüppelberg (1999).

In this paper we examine whether ρ_{SST} is a coherent multiperiod measure of risk in the sense of Artzner et al. (1999) and Artzner et al. (2004). We provide a simple example which shows that ρ_{SST} does not satisfy the monotonicity axiom. In fact, ρ_{SST} is too conservative. There are situations where the company is allowed to give away a profitable non-risky part of its asset–liability portfolio while reducing its target capital. This is unsatisfactory from the regulatory point of view. We thus find the greatest coherent multiperiod risk measure $\underline{\rho_{SST}}$ which is majorized by ρ_{SST}, and propose that ρ_{SST} be replaced by $\underline{\rho_{SST}}$.

Since the actual risk measure used in the SST is a computationally simplifying approximation of ρ_{SST}, our proposal does not directly challenge the current usage of the SST. However, on the methodological level it does.

The remainder of the paper is as follows. In Section 2 we give a formal definition of the SST risk measure ρ_{SST} within the appropriate stochastic setup. In Section 3 we provide the axioms of coherence for multiperiod risk measures and check whether ρ_{SST} satisfies them. A simple example illustrates the failure of the monotonicity axiom. In Section 4, Theorem 4.1, we find the greatest coherent risk measure majorized by ρ_{SST}, which is then proposed as an alternative. We conclude with Section 5 and give an outlook to possible future enhancements of the SST risk measure. While the proof in Section 4 of

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Theorem 4.1 is elementary and self-contained, we provide in the Appendix, as a constructive application of Filipović and Kupper (2007), an alternative proof based on a convex duality argument.

2. The SST risk measure

Throughout, $T \in \mathbb{N}$ denotes an arbitrary finite time horizon. The stochastic basis is given by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $(\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_T = \mathcal{F}$. One should think of $\mathcal{F}_t$ as the information available up to time t.

For $t = 0, \ldots, T$, $L^\infty(\mathcal{F}_t)$ denotes the space of essentially bounded random variables on $(\Omega, \mathcal{F}_t, \mathbb{P})$. Further, $\mathcal{R}^\infty$ denotes the space of essentially bounded stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$ which are adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. Equalities and inequalities between random variables and stochastic processes are understood in the $\mathbb{P}$-almost sure sense. For instance, two stochastic processes $C, D \in \mathcal{R}^\infty$ are meant to be equal if for $\mathbb{P}$-almost all $\omega \in \Omega$, $C_t(\omega) = D_t(\omega)$ for all $t = 0, \ldots, T$. For any $C \in \mathcal{R}^\infty$ we write

$$\Delta C_t = C_t - C_{t-1}.$$

The expected shortfall of $X \in L^\infty(\mathcal{F})$ at level $\alpha \in (0, 1)$ is given by

$$\text{ES}_\alpha(X) := -\frac{1}{\alpha} \int_0^\alpha q_X(s) \, ds,$$

where $q_X(s) = \inf\{x \mid \mathbb{P}(X \le x) > s\}$ denotes the upper quantile-function of X. It is well-known, see e.g. Föllmer and Schied (2004), that the expected shortfall as a risk measure $\text{ES}_\alpha : L^\infty(\mathcal{F}) \to \mathbb{R}$ satisfies the respective coherence axioms (N)–(PH) in Section 3 below for $T = 1$ (indeed, we consider $C = (C_0, C_1) \mapsto \rho(C) = \text{ES}_\alpha(C_1)$).

A core figure for the current discussion on solvency capital requirements is the target capital (TC). The target capital is understood as the minimal monetary amount needed for an insurance company to be sure that the assets at the end of a year are sufficient to cover the liabilities. Sure in this context means that even in an unlikely situation (say of a 1% probability) there is on average enough capital to allow the assets and liabilities to be transferred to a third party and in addition to this, to provide a capital endowment for that third party to cover its liabilities and future capital costs. Consequently, target capital is given by the sum of the so-called 1-year risk capital (ES) which is to cover the risk emanating within a one year time horizon and a risk margin (M) which is defined as the minimal amount that allows a healthy insurer to take over the portfolio at no additional cost. In mathematical terms,

$$TC := ES + M.$$

In the technical document of the Swiss Solvency Test (Swiss Federal Office of Private Insurance, 2004) the Swiss Federal Office of Private Insurance comes forward with a proposal on how to substantiate the 1-year risk capital and the risk margin in terms of the risk bearing capital. The risk bearing capital C_t at date t is given by the difference between the prevailing market consistent value of assets and the best estimate of liabilities. In this paper, we shall assume that the process $C = (C_0, \ldots, C_T)$ of risk bearing capital is an element of $\mathcal{R}^\infty$. The 1-year risk capital is then defined as

$$ES := \text{ES}_\alpha(\Delta C_1) = C_0 + \text{ES}_\alpha(C_1),$$

where α is currently set to 1%. The risk margin is defined as the cost of future 1-year risk capital

$$M := \beta \sum_{s=2}^T \text{ES}_\alpha(\Delta C_s),$$

where $\beta > 0$ denotes the spread above the interest rate at which money can be borrowed and reinvested at no risk. This spread is specified by the supervisor, and it is currently set to 6% (see also CEIOPS (2007, I.7.72) in the technical specifications for the Quantitative Impact Study 3 of CEIOPS).

To sum up,

$$TC = ES + M$$
$$= C_0 + \text{ES}_\alpha(C_1) + \beta \sum_{s=2}^T \text{ES}_\alpha(\Delta C_s)$$
$$= C_0 + \rho_{\text{SST}}(C),$$

where

$$\rho_{\text{SST}}(C) := \text{ES}_\alpha(C_1) + \beta \sum_{s=2}^T \text{ES}_\alpha(\Delta C_s). \quad (2.1)$$

The functional $\rho_{\text{SST}} : \mathcal{R}^\infty \to \mathbb{R}$ defined in (2.1) is called Swiss Solvency Test risk measure. We note that the actual risk measure used in the SST is a computationally simplifying approximation to ρ_{SST}, see Swiss Federal Office of Private Insurance (2004). The latter nevertheless represents the methodological basis.

3. ρ_{SST} — A coherent risk measure?

According to Artzner et al. (2004), Cheridito et al. (2006) and Vogelpoth (2006), a coherent multiperiod risk measure is a functional $\rho : \mathcal{R}^\infty \to \mathbb{R}$ which satisfies the following properties:

- **(N) Normalization:** $\rho(0) = 0$
- **(M) Monotonicity:** $\rho(C) \le \rho(D)$, for all $C, D \in \mathcal{R}^\infty$ such that $C \ge D$
- **(TI) Translation Invariance:** $\rho(C + m1_{[0,T]}) = \rho(C) - m$, for all $C \in \mathcal{R}^\infty$ and $m \in \mathbb{R}$
- **(SA) Subadditivity:** $\rho(C + D) \le \rho(C) + \rho(D)$, for all $C, D \in \mathcal{R}^\infty$
- **(PH) Positive Homogeneity:** $\rho(\lambda C) = \lambda \rho(C)$, for all $C \in \mathcal{R}^\infty$ and $\lambda \ge 0$.

We now check whether these axioms are satisfied by ρ_{SST} which was introduced in (2.1). First, we observe that ρ_{SST} satisfies (PH) since the expected shortfall ES_α is positively

homogeneous. Next, from the translation invariance of the expected shortfall we derive

$$\rho_{SST}(C + m1_{[0,T]}) = ES_\alpha(C_1 + m) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s)$$

$$= m + ES_\alpha(C_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s)$$

$$= m + \rho_{SST}(C)$$

for all $C \in \mathcal{R}^\infty$ and $m \in \mathbb{R}$, whence (**TI**) holds. Further, from the subadditivity of ES_α it follows that

$$\rho_{SST}(C + D) = ES_\alpha(C_1 + D_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s + \Delta D_s)$$

$$\leq ES_\alpha(C_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s)$$

$$+ ES_\alpha(D_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta D_s)$$

$$= \rho_{SST}(C) + \rho_{SST}(D)$$

for all $C, D \in \mathcal{R}^\infty$, and hence ρ_{SST} satisfies (**SA**). In general however, ρ^{SST} lacks (**M**) and therefore it is not a coherent risk measure. Here is a counter-example.

Let $T = 2$ and consider a stochastic model that consists only of two future states $\Omega = \{\omega_1, \omega_2\}$ such that $P(\omega_1) = P(\omega_2) = \frac{1}{2}$. Suppose $\alpha < \frac{1}{2}$. We define the risk bearing capital process

$$C(\omega_1) = (0, 0, 0), \qquad C(\omega_2) = (0, 1, 0).$$

Obviously, we have $C \geq 0$. For any $s \in (0, \frac{1}{2})$ we have $q_{C_1}(s) = 0$ and $q_{\Delta C_2}(s) = -1$, and thus

$$ES_\alpha(C_1) = -\frac{1}{\alpha} \int_0^\alpha q_{C_1}(s) ds = 0 \quad \text{and}$$

$$ES_\alpha(\Delta C_2) = -\frac{1}{\alpha} \int_0^\alpha q_{\Delta C_2}(s) ds = 1.$$

Hence $\rho_{SST}(C) = \beta > 0 = \rho_{SST}(0)$ which contradicts (**M**). Thus the company should be allowed to replace C by 0 – i.e. to dispose a profitable non-risky position – while reducing the target capital. This is unsatisfactory from a regulatory point of view.

4. A coherent modification of ρ_{SST}

The preceding example suggests that the target capital for an insurance portfolio C can be reduced by disposing a profitable non-risky position of the portfolio. This suggests that ρ_{SST} is too conservative, and may be replaced by the greatest coherent multiperiod risk measure $\underline{\rho_{SST}}$ among those which are majorized by ρ_{SST}. Indeed, it is shown in Filipović and Kupper (2007) that

$$\underline{\rho_{SST}}(C) = \min_{D \leq C} \rho_{SST}(D).$$

It is thus a remarkable and useful fact that $\underline{\rho_{SST}}$ can be calculated explicitly:

Theorem 4.1. *Suppose* $\beta \leq 1$. *Then the greatest coherent multiperiod risk measure among those which are majorized by* ρ_{SST} *is*

$$\underline{\rho_{SST}}(C) = (1 - \beta)ES_\alpha(C_1) + \beta ES_\alpha(C_T), \quad C \in \mathcal{R}^\infty. \quad (4.2)$$

Proof. The proof is divided into the following 3 steps:

- First, we show that ρ_{SST} majorizes $\underline{\rho_{SST}}$.
- Second, we observe that $\underline{\rho_{SST}}$ indeed is a coherent multiperiod risk measure.
- Third, we prove that any coherent risk measure which is majorized by ρ_{SST} in turn is majorized by $\underline{\rho_{SST}}$.

1st Step. We observe

$$\rho_{SST}(C) = ES_\alpha(C_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s)$$

$$= (1 - \beta)ES_\alpha(C_1) + \beta ES_\alpha(C_1) + \beta \sum_{s=2}^{T} ES_\alpha(\Delta C_s)$$

$$\geq (1 - \beta)ES_\alpha(C_1) + \beta ES_\alpha\left(C_1 + \sum_{s=2}^{T} \Delta C_s\right)$$

$$= (1 - \beta)ES_\alpha(C_1) + \beta ES_\alpha(C_T), \quad (4.3)$$

for all $C \in \mathcal{R}^\infty$, where the inequality in (4.3) follows from the subadditivity of ES_α. Thus, ρ_{SST} majorizes the functional given in (4.2).

2nd Step. From the respective properties of the expected shortfall ES_α we immediately derive that the functional given in Eq. (4.2) is a coherent multiperiod risk measure.

3rd Step. First, we consider the case of $T = 1$ and assume that ρ is a coherent risk measure which is majorized by ρ_{SST}. We have to show that ρ is also majorized by the coherent risk measure given in (4.2). To this end, we observe for all $C \in \mathcal{R}^\infty$,

$$\rho(C) \leq \rho_{SST}(C) = ES_\alpha(C_1) + \beta \sum_{s=2}^{1} ES_\alpha(\Delta C_s)$$

$$= ES_\alpha(C_1)$$

$$= (1 - \beta)ES_\alpha(C_1) + \beta ES_\alpha(C_1). \quad (4.4)$$

The functional in (4.4) is the coherent risk measure given in (4.2) for $T = 1$ and hence, the assertion is verified.

Now let $T = 2$ and ρ be a coherent risk measure majorized by ρ_{SST}, i.e.

$$\rho(C) \leq \rho_{SST}(C) = ES_\alpha(C_1) + \beta \sum_{s=2}^{2} ES_\alpha(\Delta C_s)$$

$$= ES_\alpha(C_1) + \beta ES_\alpha(\Delta C_2),$$

$$\text{for all } C \in \mathcal{R}^\infty. \quad (4.5)$$

Now we fix some $C \in \mathcal{R}^\infty$ and derive from the subadditivity of ρ

$$\rho(C) = \rho(C_0 1_{\{0\}} + C_1 1_{\{1\}} + C_2 1_{\{2\}})$$

$$\leq \rho(C_0 1_{\{0\}}) + \rho(C_1 1_{\{1\}}) + \rho(C_2 1_{\{2\}}). \quad (4.6)$$

Successive application of (4.5) to the three summands in (4.6)

yields

$$\rho(C_0 1_{\{0\}}) \leq \mathrm{ES}_\alpha(0) + \beta \mathrm{ES}_\alpha(\Delta 0) = 0,$$
$$\rho(C_1 1_{\{1\}}) \leq \mathrm{ES}_\alpha(C_1) + \beta \mathrm{ES}_\alpha(-C_1) = (1 - \beta)\mathrm{ES}_\alpha(C_1)$$
$$\text{and} \quad \rho(C_2 1_{\{2\}}) \leq \mathrm{ES}_\alpha(0) + \beta \mathrm{ES}_\alpha(C_2) = \beta \mathrm{ES}_\alpha(C_2).$$

Summing up both the sides we obtain

$$\rho(C_0 1_{\{0\}}) + \rho(C_1 1_{\{1\}}) + \rho(C_2 1_{\{2\}})$$
$$\leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta \mathrm{ES}_\alpha(C_2),$$

and the assertion is proved in the case of $T = 2$.

The proof for $T \geq 3$ is by induction over T. Since we have already verified the statement in the cases of $T = 1$ and $T = 2$ we may start right off with the assumption that for $T - 1$, $\rho_{\underline{\mathrm{SST}}}$ is given by (4.2). Consider a coherent risk measure ρ such that for all $C \in \mathcal{R}^\infty$

$$\rho(C) \leq \rho_{\underline{\mathrm{SST}}}(C) = \mathrm{ES}_\alpha(C_1) + \beta \sum_{s=2}^{T} \mathrm{ES}_\alpha(\Delta C_s).$$

We fix some $C \in \mathcal{R}^\infty$, and we decompose it as

$$C = C^{T-1} + \Delta C_T 1_{\{T\}},$$

where

$$C^{T-1} := C 1_{[0, T-1]} + C_{T-1} 1_{\{T\}}$$

is the process C stopped at time $T - 1$. Subadditivity of ρ yields

$$\rho(C) = \rho(C^{T-1} + \Delta C_T 1_{\{T\}})$$
$$\leq \rho(C^{T-1}) + \rho(\Delta C_T 1_{\{T\}}). \qquad (4.7)$$

As for the first summand in Eq. (4.7), we have

$$\rho(C^{T-1}) \leq \rho_{\underline{\mathrm{SST}}}(C^{T-1})$$
$$= \mathrm{ES}_\alpha(C_1) + \beta \sum_{s=2}^{T-1} \mathrm{ES}_\alpha(\Delta C_s)$$
$$\quad + \beta \mathrm{ES}_\alpha(C_{T-1} - C_{T-1})$$
$$= \mathrm{ES}_\alpha(C_1) + \beta \sum_{s=2}^{T-1} \mathrm{ES}_\alpha(\Delta C_s) + \beta \mathrm{ES}_\alpha(0)$$
$$= \mathrm{ES}_\alpha(C_1) + \beta \sum_{s=2}^{T-1} \mathrm{ES}_\alpha(\Delta C_s).$$

In view of the induction hypothesis for $T - 1$ we conclude that

$$\rho(C^{T-1}) \leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta \mathrm{ES}_\alpha(C_{T-1}).$$

As for the second summand in (4.7), we have

$$\rho(\Delta C_T 1_{\{T\}}) \leq \rho_{\underline{\mathrm{SST}}}(\Delta C_T 1_{\{T\}})$$
$$= \mathrm{ES}_\alpha(0) + \beta \sum_{s=2}^{T-1} \mathrm{ES}_\alpha(\Delta 0) + \beta \mathrm{ES}_\alpha(\Delta C_T - 0)$$
$$= \beta \mathrm{ES}_\alpha(\Delta C_T).$$

Combining these two estimates with (4.7) yields

$$\rho(C) \leq \rho(C^{T-1}) + \rho(\Delta C_T 1_{\{T\}})$$
$$\leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta \mathrm{ES}_\alpha(C_{T-1}) + \beta \mathrm{ES}_\alpha(\Delta C_T)$$
$$\leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta(\mathrm{ES}_\alpha(C_{T-1}) + \mathrm{ES}_\alpha(\Delta C_T)).$$
$$(4.8)$$

It remains to prove that (4.8) implies

$$\rho(C) \leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta \mathrm{ES}_\alpha(C_T).$$

To this end, note that the process $C \in \mathcal{R}^\infty$ was arbitrary. Thus, the inequality in (4.8) is valid for all processes in $\mathcal{R}^\infty$ and we therefore estimate the summands of the decomposition

$$\rho(C) = \rho(C 1_{[0, T-2]} + C_{T-1} 1_{\{T-1\}} + C_T 1_{\{T\}})$$
$$\leq \rho(C 1_{[0, T-2]}) + \rho(C_{T-1} 1_{\{T-1\}}) + \rho(C_T 1_{\{T\}}) \quad (4.9)$$

by means of Eq. (4.8):

$$\rho(C 1_{[0, T-2]}) \leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta(\mathrm{ES}_\alpha(0) + \mathrm{ES}_\alpha(\Delta 0))$$
$$= (1 - \beta)\mathrm{ES}_\alpha(C_1),$$
$$\rho(C_{T-1} 1_{\{T-1\}}) \leq \rho((\mathrm{ess.inf}\, C_{T-1}) 1_{\{T-1\}})$$
$$\leq (1 - \beta)\mathrm{ES}_\alpha(0) + \beta(\mathrm{ES}_\alpha(\mathrm{ess.inf}\, C_{T-1})$$
$$\quad + \mathrm{ES}_\alpha(-\mathrm{ess.inf}\, C_{T-1}))$$
$$= 0 \quad \text{and}$$
$$\rho(C_T 1_{\{T\}}) \leq (1 - \beta)\mathrm{ES}_\alpha(0) + \beta(\mathrm{ES}_\alpha(0) + \mathrm{ES}_\alpha(C_T - 0))$$
$$= \beta \mathrm{ES}_\alpha(C_T).$$

Plugging these estimates into (4.9) finally yields

$$\rho(C) \leq \rho(C 1_{[0, T-2]}) + \rho(C_{T-1} 1_{\{T-1\}}) + \rho(C_T 1_{\{T\}})$$
$$\leq (1 - \beta)\mathrm{ES}_\alpha(C_1) + \beta \mathrm{ES}_\alpha(C_T)$$

and the proof is completed for $T \geq 3$. □

5. Conclusion

In this paper, we briefly outlined the current SST-approach towards the quantification of solvency capital requirements by means of the multiperiod risk measure ρ_{SST}. We checked whether ρ_{SST} satisfies the axioms of coherence given by Artzner et al. (1999) and Artzner et al. (2004), and showed that ρ_{SST} lacks monotonicity in general. We then proposed to replace ρ_{SST} by the greatest coherent risk measure $\rho_{\underline{\mathrm{SST}}}$ which is majorized by ρ_{SST}. Our main result (Theorem 4.1) is a closed form expression for $\rho_{\underline{\mathrm{SST}}}$.

A sound risk assessment within a dynamic multiperiod framework requires consistency over time. So far, such dynamic time-consistency aspects (see Artzner et al. (2004), Cheridito et al. (2006)) have not been taken into account in the SST. In Vogelpoth (2006), we discuss dynamic time-consistency in the context of the SST, and a satisfactory solution which masters the arising difficulties is the subject of on-going research.

Appendix. An alternative proof of Theorem 4.1

In this section we provide an alternative proof of Theorem 4.1 based on a convex duality argument, which builds on the findings of Filipović and Kupper (2007). In fact, we obtain an extension of Theorem 4.1, see Remark A.1.

However, this part is not self-contained. For the terminology and background in convex analysis the reader is referred to Filipović and Kupper (2007).

In the following we need no restrictions on $\beta \geq 0$. To simplify the subsequent calculations, we write

$$\rho_{\text{SST}}(C) = \rho(C_1, \Delta C_2, \ldots, \Delta C_T)$$

for

$$\rho(X_1, \ldots, X_T) := \text{ES}_\alpha(X_1) + \beta \sum_{t=2}^{T} \text{ES}_\alpha(X_t).$$

We introduce the spaces

$$E := L^\infty(\mathcal{F}_1) \times \cdots \times L^\infty(\mathcal{F}_T),$$
$$E^* := L^1(\mathcal{F}_1) \times \cdots \times L^1(\mathcal{F}_T),$$

and we endow E with the weak topology $\sigma(E, E^*)$, which makes (E, E^*) a dual pair. It is well-known, see e.g. Föllmer and Schied (2004), that $\rho : E \to \mathbb{R}$ is a lower semicontinuous convex function. We write $\langle \mu, X \rangle = \sum_t \mathbb{E}[\mu_t X_t]$ for the duality pairing of $\mu \in E^*$ and $X \in E$, and follow the notional convention $\mu_{T+1} := 0$. Trivially, we then have

$$\langle \mu, X \rangle = \sum_{s=1}^{T} \mathbb{E}\left[\sum_{t=s}^{T} (\mu_t - \mu_{t+1}) X_s \right]$$
$$= \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{E}\left[\mu_t - \mu_{t+1} \mid \mathcal{F}_t \right] \sum_{s=1}^{t} X_s \right].$$

We endow the model space E with the partial order

$$X \succeq Y \Leftrightarrow X - Y \in \mathcal{P}$$

implied by the convex cone

$$\mathcal{P} = \left\{ X \in E \;\middle|\; \sum_{s=1}^{t} X_s \geq 0 \, \forall t \leq T \right\}.$$

The polar cone of $\mathcal{P}$ is

$$\mathcal{P}^* := \{ \mu \in E^* \mid \langle \mu, X \rangle \leq 0 \; \forall X \in \mathcal{P} \}$$
$$= \{ \mu \in E^* \mid \mathbb{E}\left[\mu_t - \mu_{t+1} \mid \mathcal{F}_t \right] \leq 0 \, \forall t \leq T \}.$$

Obviously, $C \geq D$, for $C, D \in \mathcal{R}^\infty$, if and only if $C_0 \geq D_0$ and

$$(C_1, \Delta C_2, \ldots, \Delta C_T) \succeq (D_1, \Delta D_2, \ldots, \Delta D_T).$$

Hence $\rho_{\text{SST}}(C) = \underline{\rho}(C_1, \Delta C_2, \ldots, \Delta C_T)$ where $\underline{\rho}$ is the greatest lower semicontinuous $\mathcal{P}$-monotone coherent risk measure on E majorized by ρ. From Filipović and Kupper (2007) we know that

$$\underline{\rho} = (\rho^* + \delta(\cdot \mid \mathcal{P}^*))^*,$$

where $\cdot^*$ denotes convex conjugation, and $\delta(\cdot \mid \mathcal{P}^*)$ the convex indicator function of $\mathcal{P}^*$. Since ρ is positively homogeneous, we have $\rho^* = \delta(\cdot \mid \text{dom}\rho^*)$, where $\text{dom}\rho^*$ denotes the domain of ρ^*, and hence

$$\underline{\rho}(X) = \sup_{\mu \in \text{dom}\rho^* \cap \mathcal{P}^*} \langle \mu, X \rangle. \tag{A.10}$$

We thus next have to calculate ρ^*. For $X_t \in L^\infty(\mathcal{F}_t)$ convex duality yields

$$\text{ES}_\alpha(X_t) = \sup_{\mu \in \mathcal{M}_t} \mathbb{E}[\mu X_t] = \delta^*(X_t \mid \mathcal{M}_t)$$

for

$$\mathcal{M}_t := \{ \mu \in L^1(\mathcal{F}_t) \mid \mathbb{E}[\mu] = -1 \quad \text{and} \quad -1/\alpha \leq \mu \leq 0 \}.$$

Since

$$\beta \delta^*(X_t \mid \mathcal{M}_t) = \beta \sup_{\mu \in \mathcal{M}_t} \mathbb{E}[\mu X_t]$$
$$= \delta^*(\beta X_t \mid \mathcal{M}_t) = \delta^*(X_t \mid \beta \mathcal{M}_t),$$

we conclude that

$$\rho(X) = \sum_{t=1}^{T} \delta^*(X_t \mid \mathcal{N}_t),$$

where

$$\mathcal{N}_t := \begin{cases} \mathcal{M}_1, & t = 1 \\ \beta \mathcal{M}_t, & t \geq 2. \end{cases}$$

Hence

$$\rho^*(\mu) = \sup_{X \in E} \sum_{t=1}^{T} \left(\mathbb{E}[\mu_t X_t] - \delta^*(X_t \mid \mathcal{N}_t) \right)$$
$$= \sum_{t=1}^{T} \sup_{X_t \in L^\infty(\mathcal{F}_t)} \left(\mathbb{E}[\mu_t X_t] - \delta^*(X_t \mid \mathcal{N}_t) \right)$$
$$= \sum_{t=1}^{T} \delta(\mu_t \mid \mathcal{N}_t)$$

and thus

$$\text{dom}\rho^* = \mathcal{N}_1 \times \cdots \times \mathcal{N}_T.$$

Consequently, the following properties are equivalent:

(i) $\mu \in \text{dom}\rho^* \cap \mathcal{P}^*$
(ii) $\mu \in \mathcal{N}_1 \times \cdots \times \mathcal{N}_T$ and $\mathbb{E}[\mu_t - \mu_{t+1} \mid \mathcal{F}_t] \leq 0$ for all t
(iii) $\mu_1 \in \mathcal{M}_1$, $\mu_1 \leq \mathbb{E}[\mu_2 \mid \mathcal{F}_1]$, $\beta \leq 1$ and $\mu_t = \beta \mathbb{E}[\nu_2 \mid \mathcal{F}_t]$ for all $t \geq 2$, for some $\nu_2 \in \mathcal{M}_T$
(iv) $\beta \leq 1$, $\mu_1 = (1 - \beta)\nu_1 + \beta \mathbb{E}[\nu_2 \mid \mathcal{F}_1]$ and $\mu_t = \beta \mathbb{E}[\nu_2 \mid \mathcal{F}_t]$ for all $t \geq 2$, for some $\nu_1 \in \mathcal{M}_1$, $\nu_2 \in \mathcal{M}_T$.

In particular, $\beta > 1$ implies $\text{dom}\rho^* \cap \mathcal{P}^* = \emptyset$. Combining the above with (A.10) we infer $\underline{\rho}(X) = \sup \emptyset = -\infty$ if $\beta > 1$, and

$$\underline{\rho}(X) = \sup_{\nu_1 \in \mathcal{M}_1, \nu_2 \in \mathcal{M}_T} \left(\mathbb{E}[((1 - \beta)\nu_1 + \beta\nu_2)X_1] \right.$$
$$\left. + \sum_{t=2}^{T} \mathbb{E}[\beta \nu_2 X_t] \right)$$
$$= (1 - \beta) \sup_{\nu_1 \in \mathcal{M}_1} \mathbb{E}[\nu_1 X_1] + \beta \sup_{\nu_2 \in \mathcal{M}_T} \mathbb{E}\left[\nu_2 \sum_{t=1}^{T} X_t \right]$$
$$= (1 - \beta)\text{ES}_\alpha(X_1) + \beta \text{ES}_\alpha \left(\sum_{t=1}^{T} X_t \right)$$

if $\beta \leq 1$. Whence Theorem 4.1 follows.

Remark A.1. In the above proof one can literally replace E and E^* by $L^p(\mathcal{F}_1) \times \cdots \times L^p(\mathcal{F}_T)$ and $L^q(\mathcal{F}_1) \times \cdots \times L^q(\mathcal{F}_T)$, for $p \in [1, \infty]$ and $q = \frac{p}{p-1}$, respectively, which extends the scope of Theorem 4.1.

Moreover, we conclude that $\beta > 1$ (an unrealistic spread) implies that $\rho_{\mathrm{SST}} = -\infty$, which completes the statement in Theorem 4.1.

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